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Unconditional Convergence of Abstract Splines

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1. INTRODUCTION

Let X and Y be real Hilbert spaces, and let $L(X, Y)$ denote, as usual, the space of all continuous linear operators from X into Y. For $\Lambda \subset X$, we say that an element $\sigma \in X$ Λ-interpolates a given $x \in X$ if

$$(\sigma - x) \in \ker \Lambda := \Lambda \;.$$

The set of all elements which interpolate x in this sense will be denoted by $[x]$.

A projection $p \in L(X, X)$ is called an *interpolation projection* if

$$\ker p = \ker \Lambda,$$

and the set of all such interpolation projections is denoted by $\mathscr{P}(\Lambda)$.

Suppose there is another Hilbert space Z, and an operator $T \in L(X, Z)$ which satisfies

$$\begin{aligned} &\ker T \cap \ker \Lambda = \{0\}, \\ &\text{range } T \text{ is closed}, \qquad\qquad (\exists) \\ &\dim\{\ker T\} = q < \infty. \end{aligned}$$

Then, it is known (cf. [3]) that the minimization problem

$$\inf\{\|Ty\| \colon y \in [x]\} \tag{1}$$

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has a unique solution, which we will denote by σ_x (σ_x will be called a *spline*.) It is easy to see that the mapping

$$x \longmapsto \sigma_x$$

determines an interpolation projection. It is called a spline projection.

2. Convergence of Interpolation Projections

Suppose we have a sequence of sets $\Lambda_n \subset X$, and the associated sequence of sets $\mathscr{P}(\Lambda_n) \subset L(X, X)$. We are interested in studying the convergence

$$p_n \to 1, \qquad \text{with} \quad p_n \in \mathscr{P}(\Lambda_n),$$

where 1 denotes the identity operator on X. We recall the following well-known (cf. [9]) consequence of the uniform boundedness principle and the Lebesgue inequality $\|x - p_n x\| \leqslant \|1 - p_n\| \cdot \operatorname{dist}(x; \text{range } p_n)$.

PROPOSITION 1. *An arbitrary sequence of projections p_n in $L(X, X)$ converges to 1 iff*

$$\operatorname{dist}(x; \text{range } p_n) \to 0 \text{ as } n \to \infty, \qquad \forall x \in X, \tag{2}$$

and

$$\|p_n\| \leqslant C \qquad \text{for all } n \text{ (i.e., } \{p_n\} \text{ are uniformly bounded in norm).} \tag{3}$$

As a simple modification of Proposition 1, we have

PROPOSITION 2. *For any sequence $\{p_n\}$ of interpolation projections, where $p_n \in P(\Lambda_n)$, the conditions*

$$\|p_n\| \leqslant C \qquad \text{for all } n, \tag{4}$$

$$\operatorname{dist}(x; \operatorname{span} \Lambda_n) \to 0, \qquad \forall x \in X, \tag{C}$$

imply the weak convergence

$$p_n \xrightarrow{w} 1. \tag{5}$$

Proof. Since $\ker p_n = \ker \Lambda_n$, the following implications prove the result:

$$\left\{\begin{matrix} \|p_n\| \leqslant C \\ \operatorname{dist}(x; \operatorname{span} \Lambda_n) \to 0 \\ \forall x \in X \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} \|p_n\| \leqslant C \\ \operatorname{dist}(x; \ker\ p_n) \to 0 \\ \forall x \in X \end{matrix}\right\}$$

$$\Leftrightarrow \left\{\begin{matrix} \|p_n^*\| \leqslant C \\ \operatorname{dist}(x; \operatorname{range} p_n^*) \to 0 \\ \forall x \in X \end{matrix}\right\},$$

$\Leftrightarrow p_n^* \to 1 \Rightarrow p_n \to^w 1$, where p_n^* denotes the adjoint operator associated with p_n. ∎

Remark. It is easy to see that the last implication in the proof of Proposition 2 *cannot* be reversed. Thus, condition (C) is not necessary in general. On the other hand, if

$$\ker \Lambda_n \supset \ker \Lambda_{n+1} \qquad \text{for all } n, \tag{6}$$

then (5) implies (C). To see this, suppose that (5) is valid. Then, for every $x \in X$, $p_n^* x \to^w x$. Hence (cf. [11, p. 120]), some sequence (y_n) with y_n in the convex hull of $(p_m^* x)_{m=1}^n$ converges strongly to x. But then if also (6) holds, we have $y_n \in \operatorname{span} \Lambda_n$, hence the $\lim_{n\to\infty} \operatorname{dist}(x, \operatorname{span} \Lambda_n)$ is zero.

3. Convergence of Spline Projections

Suppose that we have a sequence of sets $\Lambda_n \subset X$ such that conditions (∃) hold for all n, i.e.,

$$\ker \Lambda_n \cap \ker T = \{0\}, \qquad \forall n, \tag{7}$$

$$\operatorname{range} T \text{ is closed}, \tag{8}$$

$$\dim \ker T = q < \infty. \tag{9}$$

Then, we can consider the sequence of spline projections p_n. By definition, each p_n has the minimum norm property

$$\|T p_n x\| \leqslant \|Tx\|. \tag{10}$$

LEMMA 1. *Suppose conditions* (7), (8), (9) *and* (C) *hold. Then the sequence of norms* $\|p_n\|$ *is bounded.*

Proof. Let P and Q be the orthogonal projection onto $\ker T$ and $\ker^\perp T$, respectively. Then, according to the representation

$$X = \ker T \oplus \ker^\perp T,$$

every element $x \in X$ can be written in the form

$$x = Px + Qx, \qquad \text{where} \quad Px \in \ker T, \quad Qx \in \ker T.$$

The restriction of the operator T to $\ker T$ is an invertible operator on the range of T, and since the range of T is closed, this inverse operator can be extended to the whole space Z. (This extension will be denoted T^{-1}.) Therefore, we have

$$\|Qp_n x\| = \|T^{-1} T p_n x\| \leqslant \|T^{-1}\| \, \|Tp_n x\| \leqslant \|T^{-1}\| \, \|Tx\|,$$

the last inequality following because of the minimum norm property (10).

Now, we have to prove that also the sequence $\|Pp_n\|$ is bounded.

Let the sequence $s_1, \ldots, s_q$ form a normalized basis for the finite-dimensional space $\ker T$, i.e., $\|s_k\| = 1$. Condition (C) allows us to find a sequence $f_{n,j} \in \operatorname{span} A_n$, such that $f_{n,j} \to^{s} s_j$ (strong convergence) for all $j = 1, \ldots, q$. In particular, for sufficiently large n we have

$$\|s_j - f_{n,j}\| < \frac{1}{q} \qquad (\text{for all } j = 1, \ldots, q),$$

But $Ps_j = s_j$, hence

$$\|s_j - Pf_{n,j}\| = \|P(s_j - f_{n,j})\| \leqslant \|s_j - f_{n,j}\| < \frac{1}{q},$$

and

$$\sum_{j=1}^{q} \|s_j - Pf_{n,j}\| < 1.$$

Therefore (cf. [7, p. 197]), the elements $Pf_{n,j}$ form a basis in $\ker T$. Let

$$\psi_{n,1} := f_{n,1}; \qquad \varphi_{n,1} := \frac{\psi_{n,1}}{\|P\psi_{n,1}\|},$$

$$\psi_{n,2} := f_{n,2} - (Pf_{n,2}, P\varphi_{n,1})\, \varphi_{n,1}; \qquad \varphi_{n,2} := \frac{\psi_{n,2}}{\|P\psi_{n,2}\|},$$

etc.

$$\psi_{n,k} := f_{n,k} - (Pf_{n,k}, P\varphi_{n,k-1})\, \varphi_{n,k-1} - \cdots - (Pf_{n,k}, P\varphi_{n,1})\, \varphi_{n,1};$$

$$\varphi_{n,k} := \frac{\psi_{n,k}}{\|P\psi_{n,k}\|}$$

for all $k = 3, \ldots, q$.

One can observe that all the elements $f_{n,j}$; $\psi_{n,j}$; $\varphi_{n,j}$ converge to some elements s_j, ψ_j, φ_j in $\ker T$, and so do their projections $Pf_{n,j}$; $P\psi_{n,j}$; $P\varphi_{n,j}$. Moreover, since f_j, ψ_j, φ_j are elements of $\ker T$, their corresponding projections converge to the same elements, and we obtain the uniform boundedness of the sequence $\|\varphi_{n,k}\|$.

At the same time, $P\varphi_{n,k}$ is just the Gram–Schmidt basis according to the basis $Pf_{n,k}$. Hence, the elements $Pp_n x$ can be expressed in the form

$$Pp_n x = \sum_{j=1}^{q} (Pp_n x, P\varphi_{n,j}) P\varphi_{n,j}.$$

To prove the lemma, we have to prove that the absolute values $|(Pp_n x, P\varphi_{n,j})|$ are bounded uniformly by n.

Indeed, we have

$$\begin{aligned}(Pp_n x, P\varphi_{n,j}) &= (p_n x, P\varphi_{nj}) \\ &= (p_n x, \varphi_{n,j}) - (p_n x, Q\varphi_{nj}) \\ &= (p_n x, \varphi_{n,j}) - (Qp_n x, \varphi_{n,j}).\end{aligned}$$

Previously, we proved that $\|Qp_n x\|$; $\|\varphi_{n,j}\|$; $\|P\varphi_{n,j}\|$ are uniformly bounded, so that $|(Qp_n x, \varphi_{n,j})|$ are also uniformly bounded. On the other hand, $\varphi_{n,j} \in \operatorname{span} \Lambda_n$, $x - p_n x \in \ker \Lambda_n$, and thus

$$|(p_n x, \varphi_{n,j})| = |(x, \varphi_{n,j})| \leqslant \|x\| \, \|\varphi_{n,j}\|,$$

which is again bounded, uniformly in n.

To put all these results together, we have proved that the elements

$$\|p_n x\| = \|Pp_n x + Qp_n x\| \leqslant \|Qp_n x\| + \sum_{j=1}^{k} |(Pp_n x, P\varphi_{n,j})| \, \|P\varphi_{n,j}\|$$

are uniformly bounded, which proves the lemma. ∎

THEOREM 1. *Under the conditions of Lemma* 1, *spline projections converge strongly to the identity operator.*

Proof. Lemma 1 and Proposition 2 give

$$p_n \xrightarrow{w} 1.$$

Hence, since

$$Tp_n \xrightarrow{w} T, \tag{11}$$

we have

$$\|Tx\| \leqslant \underline{\lim} \|Tp_n x\|.$$

On the other hand, from (10), we obtain that

$$\|Tp_n x\| \leqslant \|Tx\|$$

and thus

$$\|Tp_n x\| \to \|Tx\|.$$

Combining the last result with (11), we have the strong convergence

$$Tp_n \to T.$$

Thus, using the notations of Lemma 1, we obtain

$$Qp_n x = T^{-} Tp_n \to T^{-} Tx = Qx.$$

At the same time, the sequence $Pp_n x$ belongs to the finite-dimensional space $\ker T$, where weak convergence implies strong convergence. Hence, $Pp_n \to Px$, and thus

$$p_n x = Pp_n x + Qp_n x \to Px + Qx = x. \quad ∎$$

4. Examples

In this section, we shall investigate condition (C) in some particular cases. First, we shall prove

PROPOSITION 3. *In an arbitrary Hilbert space X, condition* (C) *holds iff the conditions*

$$z_n \in \ker \Lambda_n, \tag{12}$$

$$\|z_n\| \leqslant 1 \tag{13}$$

imply

$$z_n \xrightarrow{w} 0. \tag{14}$$

Proof. Let P_n be the orthogonal projection on span Λ_n, and let (C) hold. Then, $\|P_n\| = 1$ and by Proposition 1

$$P_n \xrightarrow{s} 1.$$

Suppose a sequence $\{z_n\}$ satisfies (12) and (13). Then, $(1-P_n)z_n = z_n$ and for every $x \in X$,

$$|(x, z_n)| = |(x, (1-P_n)z_n)| = |(x - P_n x, z_n)| \leqslant \|x - P_n x\| \to 0.$$

Conversely, let (12) and (13) imply (14). Then, for a given $x \in X$, the elements $z_n = (x - P_n x)/\|x\|$ satisfy (12) and (13). Thus, $P_n x \to^w x$. But, the P_n's are orthogonal projections, and thus $P_n x \to x$. ∎

Let Ω be a compact domain in $\mathbb{R}^s$, and let $H_2^m(\Omega)$ be the Sobolev space of function with the finite norm

$$\|x(t)\|^2 = \sum_{|\alpha| \leqslant m} \|\mathfrak{D}^\alpha x(t)\|^2_{L_2(\Omega)},$$

where

$$\mathfrak{D}^\alpha = \frac{\partial^{\alpha_1 + \cdots + \alpha_s}}{\partial t_1^{\alpha_1} \cdots \partial t^{\alpha_s}}; \qquad t = (t_1, ..., t_s) \in \Omega;$$

$$\alpha = (\alpha_1, ..., \alpha_s); \qquad |\alpha| = \alpha_1 + \cdots + \alpha_s.$$

To ensure the compactness of the imbedding $H_2^m(\Omega) \subset C(\Omega)$, we let $2m > s$. We will also need the subset $S \subset H_2^m(\Omega)$ of all point evaluations, i.e.,

$\delta \in S$ iff there exists a point $t \in \Omega$ such that $(\delta, x) = x(t)$ for all x.

In this case, we denote δ by δ_t. It is easy to see that S is dense in $H_2^m(\Omega)$, provided $H_2^m(\Omega) \subseteq C(\Omega)$. Indeed,

$$(x, \delta) = 0, \qquad \forall \delta \in S,$$

implies

$$x(t) = 0, \qquad \forall t \in \Omega,$$

and thus $x = 0$. Therefore, if we have to prove that $z_n(t) \to^w 0$, it is enough to prove that $z_n(t) \to 0$, $\forall t \in \Omega$.

EXAMPLE 1. Let $\Delta_n = \{t_j^{(n)}\}_{j=1}^{e(n)}$ be a set of points in Ω ($e(n)$ is some function dependent on n); and let

$$\lambda_j^{(n)} x = x(t_j^{(n)}) \colon X = H_2^m(\Omega) \qquad \text{and} \qquad \Lambda := \bigcup_{j=1}^{e(n)} \{\lambda_j^{(n)}\}.$$

PROPOSITION 4. *Condition* (C) *holds if for any point* $t \in \Omega$, *there exist* $t_n \in \Delta_n$ *such that* $t_n \to t$, *i.e.*, $\Omega = \overline{\lim}_{n\to\infty} \Delta_n$.

Proof. It is clear that

$$\ker \Lambda_n = \{x \in H_2^m(\Omega): x(t_j^{(n)}) = 0, \forall j = 1,..., e(n)\}.$$

Suppose we have a sequence $\{z_n\}$ satisfying (12) and (13). We have to prove that for every $t \in \Omega$,

$$z_n(t) \to 0.$$

Suppose $t_n \in \Delta_n$ tends to t. Then,

$$z_n(t) = z_n(t) - z_n(t_n). \tag{15}$$

On the other hand, $z_n(t)$ is uniformly bounded in $H_2^m(\Omega)$ and thus it is compact in $C(\Omega)$. This implies the equicontinuity of $z_n(t)$ and hence, the difference (15) tends to zero. ∎

EXAMPLE 2. Let $\Omega = [0, 1]$, let Δ_n be defined as in Example 1, and let

$$\lambda_j^{(n)} = \left\{\frac{1}{t_{j+1}^{(n)} - t_j^{(n)}} \int_{t_{j+1}^{(n)}}^{t_j^{(n)}} x(t)\, dt\right\}, \qquad \text{where} \quad \Lambda_n := \{\lambda_j^{(n)}\}_{j=1}^{e(n)}.$$

First, recall the following well-known property of the Steklov functions (cf. [1, p. 174]). If s is the midpoint of the interval $[a, b]$, then for every absolutely continuous function $x(t)$,

$$\left| x(s) - \frac{1}{b-a} \int_a^b x(t)\, dt \right| \leqslant \omega\left(\frac{b-a}{2}; x\right),$$

where $\omega(\cdot\,; \cdot)$ is the modulus of continuity.

PROPOSITION 5. *Condition* (C) *holds if* $[0, 1] = \overline{\lim}\, \Delta_n$.

Proof. First, define

$$\bar{\Delta}_n := \left\{s_j^{(n)} = \frac{t_{j+1}^{(n)} + t_j^{(n)}}{2}, j = 1,..., e(n) - 1\right\}.$$

Then, for every $t \in \Omega$, there exist a sequence $s_n \in \bar{\Delta}_n$ such that

$$s_n \to t.$$

Consider the sequence $\{z_n\}$ satisfying (12) and (13), i.e., $\|z_n\|$ is bounded and

$$\frac{1}{t_{j+1}^{(n)} - t_j^{(n)}} \int_{t_{j+1}^{(n)}}^{t_j^{(n)}} z_n(t)\, dt = 0, \qquad \forall j = 1,..., e(n) - 1.$$

Since for each n there exist $j = j(n)$ such that

$$s_n = \frac{t_{j+1}^{(n)} + t_j^{(n)}}{2},$$

we obtain

$$\begin{aligned} |z_n(t)| &= |z_n(t) - z_n(s_n) + z_n(s_n)| \\ &\leqslant |z_n(t) - z_n(s_n)| + \left| z_n(s_n) - \frac{1}{t_{j+1}^{(n)} - t_j^{(n)}} \int_{t_{j+1}^{(n)}}^{t_j^{(n)}} z_n(t)\, dt \right| \\ &\leqslant |z_n(t) - z_n(s_n)| + \omega \left(\frac{t_{j+1}^{(n)} - t_j^{(n)}}{2} ; z_n(t) \right). \end{aligned}$$

By the equicontinuity of $\{z_n(t)\}$, the above sum tends to 0. ∎

5. Final Remarks

As a simple application of Theorem 1, one can obtain the proof of the convergence of all examples of spline-functions given in [2], under the above-mentioned conditions on A_n.

The convergence of spline projections was proved earlier in [5, 6, 8–10] under the additional assumption

$$\ker A_n \supset \ker A_{n+1}, \tag{16}$$

which in some particular cases means that the interpolation knots are nested. On the other hand, condition (16) is not satisfied by interpolation of equidistant points, or by Chebyshev points, etc. In contrast, Theorem 1 does not contain this defect. On the other hand, the remark following Proposition 2 shows that under assumption (16), any other condition on A_n (such as $\bigcap \ker A_n = \varnothing$ or $\bigcup \operatorname{span} A_n$ being dense in X) implies (C).

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